# Coupled Pendula with Cobra3

LEP 1.3.25 -11

## Related topics

Spiral spring, gravity pendulum, spring constant, torsional vibration, torque, beat, angular velocity, angular acceleration, characteristic frequency.

## Principle

Two equal gravity pendula with a particular characteristic frequency are coupled by a “soft” spiral spring. The amplitudes of both pendula are recorded as a function of time for various vibrational modes and different coupling factors using a y/t recorder. The coupling factors are determined by different methods.

## Equipment

| | | |
|---|---|---|
| Pendulum w. recorder connection | 02816.00 | 2 |
| Helical spring, 3 N/m | 02220.00 | 1 |
| Rod with hook | 02051.00 | 1 |
| Weight holder f. slotted weights | 02204.00 | 1 |
| Slotted weight, 10 g, black | 02205.01 | 5 |
| Capacitor, 10 $\mu$F/35 V | 39105.28 | 2 |
| Cobra3 Basic Unit | 12150.00 | 1 |
| Power supply 12 V | 12151.99 | 1 |
| Cobra3 Universal writer software | 14504.61 | 1 |
| RS 232 data cable | 14602.00 | 1 |
| Power supply 0-12 V DC/6 V, 12 V AC | 13505.93 | 1 |
| Bench clamp -PASS- | 02010.00 | 2 |
| Support rod -PASS-, square, $l$ = 630 mm | 02027.55 | 2 |
| Right angle clamp -PASS- | 02040.55 | 2 |
| Measuring tape, $l$ = 2 m | 09936.00 | 1 |
| Connecting cord, $l$ = 1000 mm, red | 07363.01 | 4 |
| Connecting cord, $l$ = 1000 mm, blue | 07363.04 | 4 |
| PC, Windows® 95 or higher | | |

Fig. 1: Experimental set-up for the measurement of the vibrational period of coupled pendula.

## Tasks

1. To determine the spring constant of the coupling spring.
2. To determine and to adjust the characteristic frequencies of the uncoupled pendula.
3. To determine the coupling factors for various coupling-lengths using
   a) the apparatus constants
   b) the angular frequencies for “-inphase” and “in opposite phase” vibration
   c) the angular frequencies of the beat mode.
4. To check the linear relation between the square of the coupling-lengths and
   a) the particular frequencies of the beat mode
   b) the square of the frequency for “in opposite phase” vibration.
5. To determine the pendulum’s characteristic frequency from the vibrational modes with coupling and to compare this with the characteristic frequency of the uncoupled pendula.

## Set-up and procedure

Before measurement can begin, the exact value of the spring constant $D_F$ of the coupling spring has to be determined. A supporting rod is fixed to the edge of table by means of a bench clamp. The spring is suspended on the rod from a hook which is attached to the supporting rod via a right angle clamp. Applying Hook's law

$$F = -D_F x$$

the spring constant $D_F$ can be calculated if the extension $x$ of the spring is measured for different slotted weights attached to the spring.

The pendula are then set up without coupling springs as shown in Fig. 1. The input sockets of the pendula are now switched in parallel to the DC-output of the power supply unit. The yellow output sockets of the pendula are connected to the Cobra3. The DC-output voltage of the power supply unit is adjusted to 10 V. For the channels CH 1 and CH 2, a value of 10 V is selected as the measuring range on the Cobra3.

To set the pendula into vibration the pendula rods are touched with the finger-tips on their upper third and simultaneously moved to and fro till the desired amplitudes have been established. In this way transverse vibrations can be avoided. In view of the subsequent experiments with coupled pendula care should be taken already at this stage to ensure that the pendula are oscillating in the same plane.

From the plotted curves the period $T_0$ is determined several times for each pendulum. The mean values of the periods, $\overline{T}_0$, of both pendula have to be identical within the limits of error. If deviations are observed, the lengths of the pendulum rods have to be adjusted. This is done by detaching the counter nut on the threaded rod of the pendulum weight, adjusting the pendulum length and manually retightening the counter nut.

For the performance of the experiments with coupled pendula, the coupling spring is fixed to the plastic sleeves on the pendulum rods at a point equidistant from the pendulum's fulcrum. Furthermore the "zero"-positions have to be readjusted. It has to be insured that there is no electric conductivity between the pendula.

The amplitudes as a function of time are to be recorded for different coupling lenghts $l$ using the following initial conditions:

A Both pendula are deflected with the same amplitude to the same side and simultaneously released. ("in- phase" vibration)

B. Both pendula are deflected with the same amplitude but in opposite directions and simultaneously released. ("in opposite phase" vibration)

C. One pendulum remains at rest. The second pendulum is deflected and released (beat mode). Here satisfactory results can only be achieved if during the preparation both pendula have been properly adjusted in such a way that they have in fact the same period $\overline{T}_0$.

In all three cases the vibrations have to be recorded for at least three or four minutes. From the plotted curves the mean values for the corresponding vibrational periods can be determined.

## Setting of the Cobra3 basic unit

— Connect the recorder outputs of the pendula to the analog inputs of the Cobra3 basic unit. The signals that are to measure in this experiment are rather slow. To reduce the sensitivity to the fast noise signals use the 10 $\mu$F capacitors at the analog inputs of the Cobra3 basic unit.

— Connect the COBRA3 Basic Unit to the computer port COM1, COM2 or to USB port (use USB to RS232 Adapter 14602.10). Start the measure program and select Cobra3 Universal Writer Gauge.

— Begin recording the measured values using the parameters given in Fig. 2.

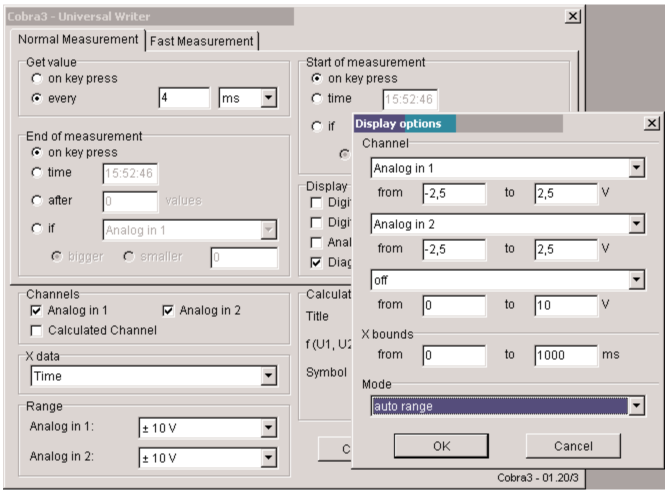


Fig. 2: Measuring parameters

## Theory and evaluation

If two gravity pendula $P_1$ and $P_2$ with the same angular characteristic frequency $\omega_0$ are coupled by a spring, for the position of rest and small angle deviation ~ due to the presence of gravity and spring-tension we have the following torques (Fig. 3):

torque due to gravity:

$$M_{s,0} = m\, g\, L \sin \phi_0 \sim m\, g\, L \phi_0 \qquad (1)$$

torque due to spring-tension:

$$M_{F,0} = -D_F x_0\, l \cos \phi_0 \sim -D_F x_0\, l$$

$D_F$ = spring constant
$x_0$ = extension of the spring
$l$ = coupling length
$m$ = pendulum mass
$L$ = pendulum lengh
$g$ = acceleration due to gravity
$\phi_0$ = angle between the vertical and the position of rest

If $P_1$ is now deflected by $\phi_1$ and $P_2$ by $\phi_2$ (see Fig. 3) and subsequently released, we have because of

$$I \ddot{\phi} = M$$

$I$ = moment of inertia of a pendulum around its fulcrum

$$I \ddot{\phi}_1 = M_1 = -mgL\phi_1 + D_F l^2 (\phi_2 - \phi_1) \quad (2)$$
$$I \ddot{\phi}_2 = M_2 = -mgL\phi_2 + D_F l^2 (\phi_2 - \phi_1)$$

Introducing the abbreviations

$$\omega_0^2 = \frac{mgL}{I} \text{ and } \Omega^2 = \frac{D_F l^2}{I} \quad (3)$$

we obtain from Eqs. (2)

$$\ddot{\phi}_1 + \omega_0^2 \phi_2 = -\Omega^2 (\phi_2 - \phi_1) \quad (4)$$
$$\ddot{\phi}_2 + \omega_0^2 \phi_2 = +\Omega^2 (\phi_2 - \phi_1)$$

At $t = 0$ the following three initial conditions are to be realized successively.

A: "inphase" vibration

$$\phi_1 = \phi_2 = \phi_A;\ \phi_1 - \phi_2 = 0$$

B: "in opposite phase" vibration

$$-\phi_1 = \phi_2 = \phi_A;\ \phi_1 - \phi_2 = 2\phi_A \quad (5)$$

C: beat mode

$$\phi_1 = \phi_A;\ \phi_2 = 0;\ \phi_1 - \phi_2 = \phi_A$$

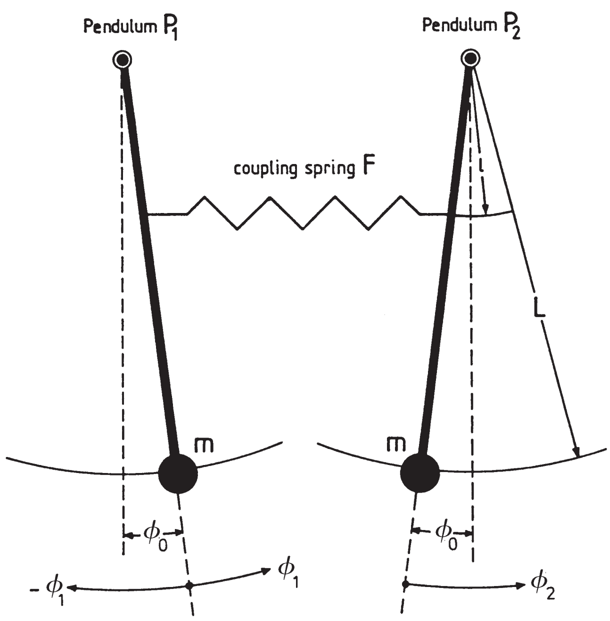


Fig. 3: Diagram of coupled pendula at rest.

The general solutions of the system of differential equations (4) with the initial conditions (5) are:

A: $\phi_1(t) = \phi_2(t) = \phi_A \cos \omega_0 t$ (6a)

B: $\phi_1(t) = \phi_A \cos \left( \sqrt{\omega_0^2 + 2\,\Omega^2}\, t \right)$ (6b)

$\phi_2(t) = -\phi_A \cos \left( \sqrt{\omega_0^2 + 2\,\Omega^2}\, t \right)$

C: $\phi_1(t) = \phi_A \cos \left( \frac{\sqrt{\omega_0^2 + 2\,\Omega^2} - \omega_0}{2} \cdot t \right)$ (6c)

$\cdot \cos \left( \frac{\sqrt{\omega_0^2 + 2\,\Omega^2} + \omega_0}{2} \cdot t \right)$

$\phi_2(t) = -\phi_A \sin \left( \frac{\sqrt{\omega_0^2 + 2\,\Omega^2} - \omega_0}{2} \cdot t \right)$

$\cdot \sin \left( \frac{\sqrt{\omega_0^2 + 2\,\Omega^2} + \omega_0}{2} \cdot t \right)$

**Comment**

A: "inphase" vibration
Both pendula vibrate inphase with the same amplitude and with the same frequency $\omega_g$. The latter is identical with the angular characteristic frequency $\omega_0$ of the uncoupled pendula.

$\omega_g = \omega_0$ (7a)

B: "in opposite phase" vibration
Both pendula vibrate with the same amplitude and with the same frequency $\omega_c$ but there is a phase-difference of $\pi$. In accordance with (3), the angular frequency

$\omega_c = \sqrt{\omega_0^2 + 2\,\Omega^2}$ (7b)

depends on the coupling length $l$.

C: Beat mode
For weak coupling, e.g. $v_0 \gg \Omega$, the angular frequency of the first factor can be expressed as follows:

$\omega_1 = \frac{\sqrt{\omega_0^2 + 2\,\Omega^2} - \omega_0}{2} \simeq \frac{\Omega^2}{2\omega_0}$ (8a)

For the angular frequency of the second factor we get:

$\omega_2 = \frac{\sqrt{\omega_0^2 + 2\,\Omega^2} + \omega_0}{2} \simeq \omega_0 + \frac{\Omega^2}{2\omega_0}$ (8b)

Subsequently we get:

$\omega_1 < \omega_2$

Fig. 4 shows the amplitudes $\phi_1(t)$ and $\phi_2(t)$ of both pendula as a function of time for the beat case and for different coupling lenghts $l$. As coupling factor we define the ratio

$K = \frac{D_F l^2}{mgL + D_F l^2}$ (9)

From Eq. (3) and Eq. (9) we get

$$K = \frac{\Omega^2}{\omega_0^2 + \Omega^2} \qquad (10)$$

The coupling factor $K$ of Eq. (10) can be calculated from the frequencies of the individual vibrational modes.

Substituting Eq. (7a) and Eq. (7b) into Eq. (10) results in

$$K = \frac{\omega_c^2 - \omega_g^2}{\omega_c^2 + \omega_g^2} \qquad (11)$$

("in opposite phase" vibration)

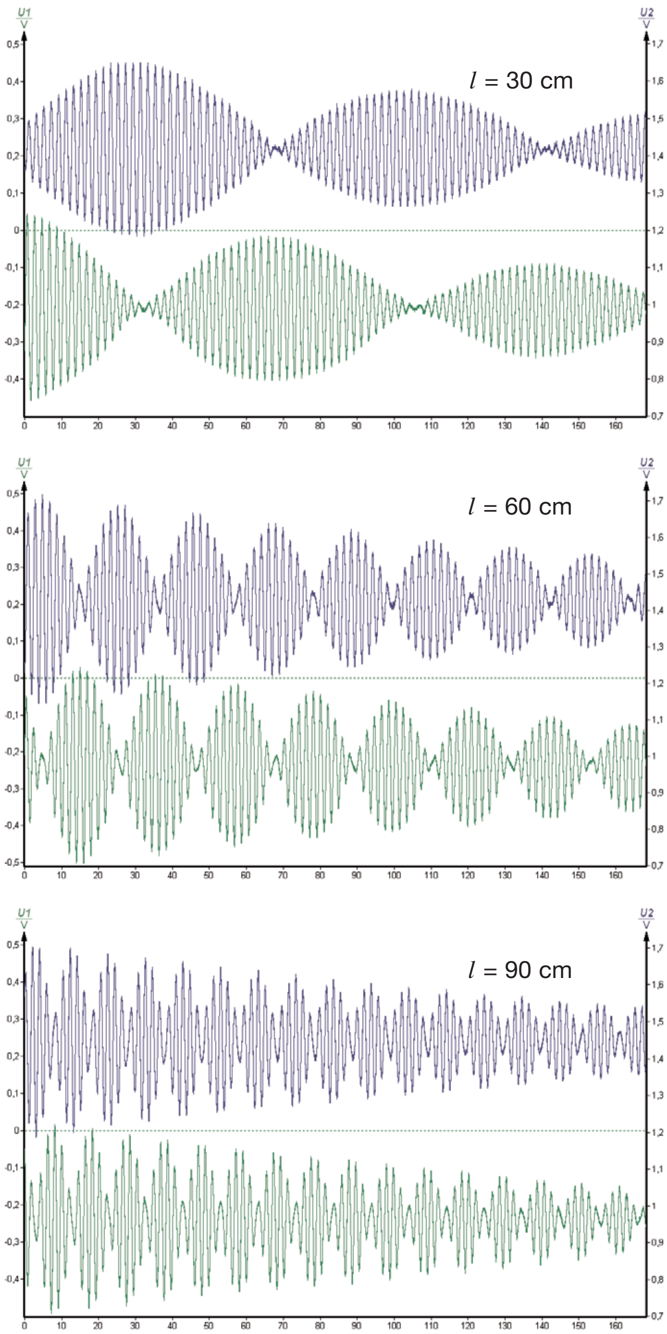


Fig. 4: Amplitude curves of the vibrations of coupled pendula in the beat mode for three different coupling lengths $l$ (30 cm, 60 cm and 90 cm) as a function of time.

Substituting Eq. (8a) and Eq. (8b) into Eq. (10) yields:

$$K = \frac{2\omega_1\omega_2}{\omega_1^2 + \omega_2} \qquad (12)$$

(beat case)

To check the influence of coupling length on the frequencies of the individual vibrational modes, we substitute Eq. (11) and Eq. (12) into Eq. (9). Then we get for the in opposite phase vibration:

$$\omega_1^2 = \frac{2D_F\omega_0^2}{mgL}\, l^2 + \omega_0^2 \qquad (13)$$

And for the beat mode:

$$\omega_1 = \omega_0 \, \frac{D_F}{2\; mgL} \; l^2 \qquad (14)$$

as well as

$$\omega_2 = \omega_0 \, \frac{D_F}{2\; mgL} \; l^2 + \omega_0 \qquad (15)$$

The measurement of the "inphase" vibration of the uncoupled pendula results in the following:

$$\overline{T}_0 = (2.036 \pm 0.003)\ \text{s}; \quad \frac{\Delta \overline{T}_0}{\overline{T}_0} \approx \pm\, 0.15\,\% \qquad (16)$$

or $$\frac{2\pi}{\overline{T}_0} = \overline{\omega}_0 = (3.083 \pm 0.005)\ \text{s}^{-1}$$

**Analysis of the measurement**

For the analysis of the results select the following parameters: In the "Analysis" / "Channel modification" window (see Fig. 5) select:

| | |
|---|---|
| source channel: | Time |
| Operation: | f := x/1000 |
| Destination channel: | overwrite / Time |
| Title: | Time_in_seconds |
| Symbol: | t |
| Unit: | s |

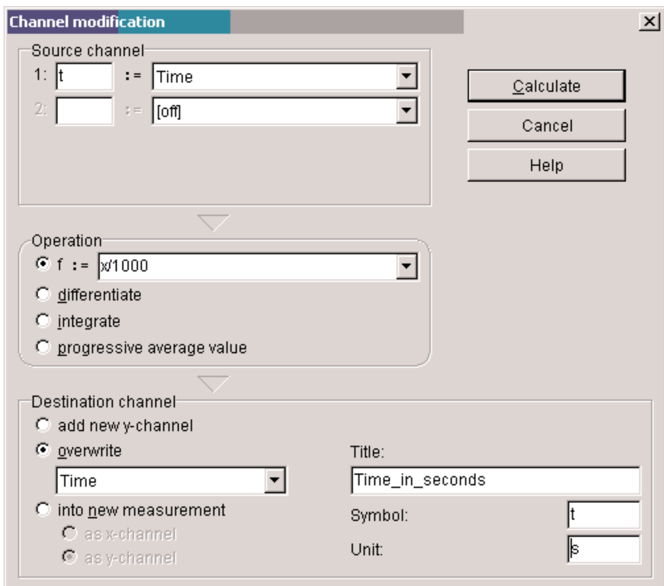


Fig. 5: Channel modification.

To determine the frequency of the in opposite phase vibration select in the "Analysis" window the option "Fourier analysis" for the measured channel (see Fig. 6):

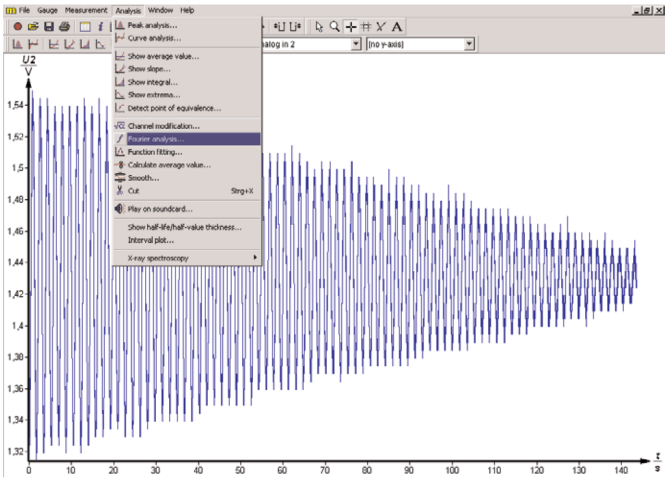

Fig. 6: Fourier analysis of the in opposite phase vibration.

With the function "Survey" determine then the frequency $f_c$ (see Fig. 7). In our example $f_c$ = 0.571 Hz for the coupling length $l$ = 80 cm and $\omega_c = 2\pi f_c$ = 3.580 1/s.

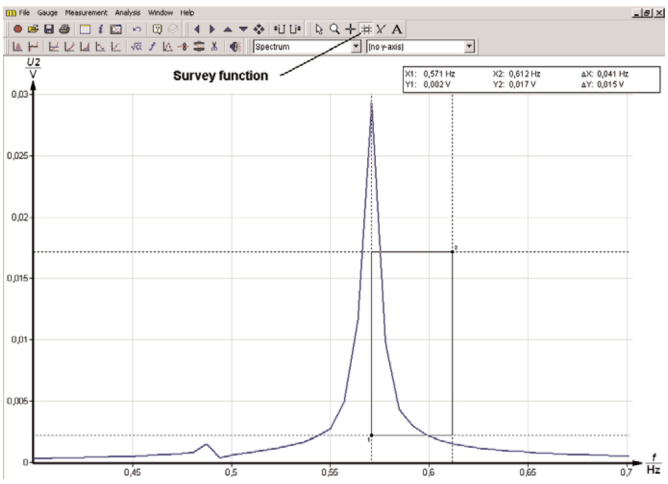


Fig. 7: Determination of the frequency of the in opposite phase vibration.

With the same procedure ("Analysis" / "Fourier analysis" / "Survey" function) determine the frequencies $\omega_1$ and $\omega_2$ for the beat case. In our example (see Fig. 8) we obtain for the coupling length $l$ = 90 cm: $f_1$ = 0.099 Hz, $\omega_1 = \pi f_1$ = 0.311 1/s and $\omega_2 = \omega_1 + 2\pi f_0$ = 3.401 1/s, with $f_0$ = 0.491 Hz.

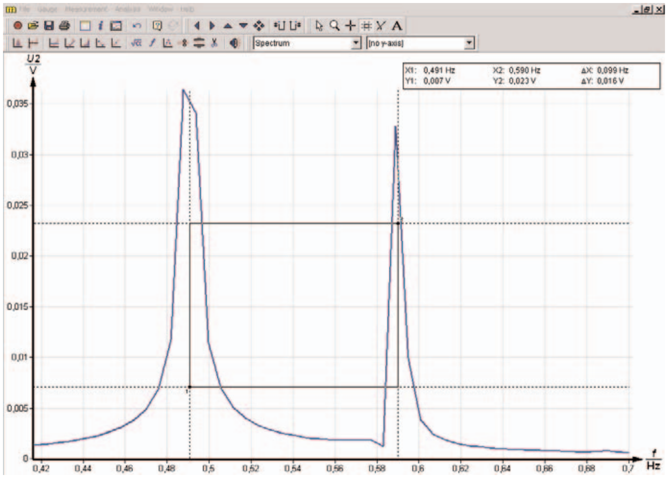

Fig. 8: Determination of the frequencies $\omega_1$ and $\omega_2$ for the beat mode.

Tab. 1 shows the mean values of the vibrational periods for different coupling lengths $l$ as well as the corresponding angular frequencies.

From the measured values of the "inphase" vibration we get

$$\overline{T}_g \triangleq (2.033 \pm 0.004)\ \text{s}; \quad \frac{\Delta \overline{T}_0}{\overline{T}_0} \approx \pm 0.2\,\%$$

or $$\frac{2\pi}{\overline{T}_0} = \overline{\omega}_0 = (3.090 \pm 0.006)\ \text{s}^{-1}$$

| $l$/m | $T_c$/s | $\frac{2\pi}{T_c} = \omega_c$/s$^{-1}$ | $T_1$/s | $\frac{\pi}{T_1} = \omega_1$/s$^{-1}$ | $T_2$/s | $\frac{2\pi}{T_2} = \omega_2$/s$^{-1}$ |
|---|---|---|---|---|---|---|
| 0.300 | 1.978 | 3.166 | 71.429 | 0.044 | 2.004 | 3.134 |
| 0.400 | 1.938 | 3.244 | 47.619 | 0.066 | 1.999 | 3.156 |
| 0.500 | 1.891 | 3.298 | 31.250 | 0.100 | 1.969 | 3.190 |
| 0.600 | 1.837 | 3.361 | 20.833 | 0.151 | 1.938 | 3.240 |
| 0.700 | 1.779 | 3.463 | 16.129 | 0.195 | 1.912 | 3.284 |
| 0.800 | 1.719 | 3.580 | 12.658 | 0.248 | 1.881 | 3.338 |
| 0.900 | 1.657 | 3.667 | 10.101 | 0.311 | 1.847 | 3.401 |

Tab. 1

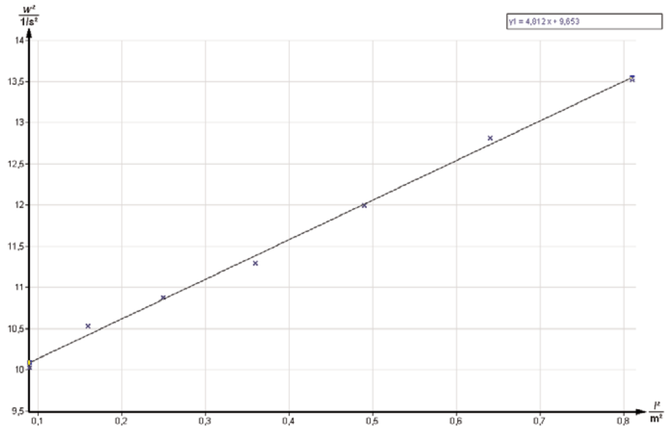


Fig. 9: Frequency of the opposite phase vibration $\omega_c^2$ as a function of the coupling length $l^2$.

We used:

$D_F$ = 3.11 N/m (measured value)

$L = L_1 = L_2$ = 101.5 cm

(distance fulcrum center of pendulum weight)

$m$ = 1 kg

(mass of pendulum rod is not included)

$g$ = 9.81 m/s

In Fig. 9 the measured values $\omega_c^2$ of Tab. 1 have been plotted versus $l^2$. From the regressive line

$$y = A + Bx$$

we obtain

$$A = (9.65 \pm 0.58)\ \text{s}^{-2};\quad \frac{\Delta A}{A} = \pm 6\,\%$$

$$B = (4.81 \pm 0.14)\ \text{s}^{-2}\ \text{m}^{-2};\quad \frac{\Delta B}{B} = \pm 3\,\%$$

Comparison with Eq. (13) gives

$$\sqrt{A} = \omega_0 = (3.106 \pm 0.093)\ \text{s}^{-1};\quad \frac{\Delta\omega_0}{\omega_0} = \pm 3\%$$

In Fig. 10 the measured values $\omega_2$ of Tab. 1 have been plotted versus $l^2$. The regression line

$$y = A + Bx$$

should confirm Eq. (15). We obtain:

$$A = \omega_0 = (3.099 \pm 0.009)\ \text{s}^{-1};\quad \frac{\Delta\omega_0}{\omega_0} = \pm 3\ \%$$

$$B = (0.374 \pm 0.015)\ \text{s}^{-1}\ \text{m}^{-2};\quad \frac{\Delta B}{B} = \pm 4\,\%$$

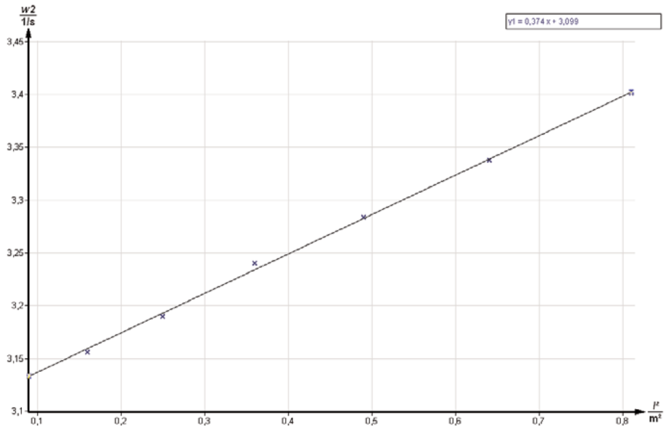


Fig. 10: Frequency $\omega_2$ (beat mode) as a function of the coupling length $l^2$.

In Fig. 11 the measured values $\omega_1$ of Tab. 1 are plotted as a function of $l^2$. The regression line $y = Bx$ through the origin confirms Eq. (14), $B$ = (0.374± 0.015) 1/s$^2$ m$^2$.

The results obtained for $\omega_0$ using three different vibrational modes for the coupled pendula are in good agreement with the angular characteristic frequency of the uncoupled pendula.

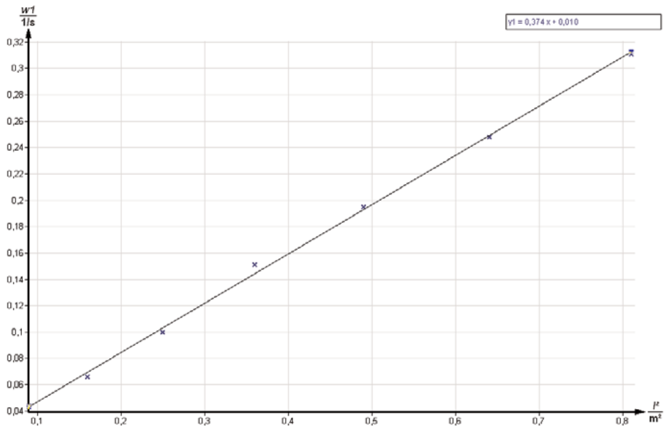


Fig. 11: Frequency $\omega_1$ (beat mode) as a function of the coupling length $l^2$.